On the use of colored hyperbolic subgroup patterns in finding covering maps for triply periodic minimal surfaces

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Abstract. It has long been recognized that group theory is an indispensable mathematical tool in many branches of science such as chemistry and physics. In chemistry, materials with particular properties need to know what structures are possible, and which are the most likely to form. This has led to studies in the theoretical aspects of description and analysis of symmetric structures such as nets, tilings, and surfaces.

Interests in crystal nets or triply periodic embeddings of graphs in three-dimensional Euclidean space stems from their fundamental relevance to condensed materials. Infinite periodic minimal surfaces are now being introduced to describe some complex structures with large cells. The crystallographic properties of these surfaces are studied from an intrinsic point of view, using operations of groups of symmetry.

Our approach takes advantage of the relation existing between these groups and those characterizing the colored tilings of the hyperbolic plane. Particularly, subgroups of triangle groups obtained from transitive colorings of the hyperbolic plane are used to give an enumeration of crystal nets.

Keywords: colored hyperbolic patterns, covering maps, triply periodic minimal surfaces, color symmetry

INTRODUCTION

It has long been recognized that group theory is an indispensable mathematical tool in many branches of science such as chemistry and physics. In chemistry, where there is an explosive growth in inorganic and organic materials chemistry, chemists interested in synthesizing materials with particular properties need to know what structures are possible, and which are the most likely to form. This has led to studies in the theoretical aspects of description and analysis of symmetric structures such as nets, tilings, and surfaces, among others.

Interest in crystal nets or triply periodic embeddings of graphs in three dimensional Euclidean space stems from their fundamental relevance to condensed materials, from atomic crystals to novel framework materials including carbon polymorphs, zeolites,
related oxide, and alumino-phosphate (AlPO) materials, imidazolate (ZIF) frameworks and metalcoordination polymeric materials. The microdomains of some liquid-crystalline phases of soft materials, including amphiphile and copolymer assemblies and derivative mesoporous solids, are also characterized by crystal nets [13].

Moreover, infinite periodic minimal surfaces are now being introduced to describe some complex structures with large cells, formed by inorganic and organic materials, which can be considered as crystals of surfaces or films [10]. Among them are the cubic crystalline structures built by amphiphilic molecules in the presence of water. The crystallographic properties of these surfaces are studied from an intrinsic point of view, using operations of groups of symmetry defined by displacements on their surface. This approach takes advantage of the relation existing between these groups and those characterizing the tilings of the hyperbolic plane.

In this study of crystalline networks, a close connection between hyperbolic non-euclidean geometry, group theory and the molecular structures is established. Hyperbolic tilings, in particular, play a vital role in the enumeration of crystal networks. These tilings whose symmetries are commensurate with Euclidean symmetries produce symmetric structures. We view these tilings as covering maps of surfaces. Some of these tilings considered have symmetry groups *642, *742 and *666.

Let \( G = *pqr \) and \( H \leq G \). First, we consider \( H \)-transitive \( n \)-colorings of the hyperbolic plane. The subgroups of the hyperbolic tiling are associated with perfect colorings of the tiling. An enumeration of these colorings associated with each subgroup is given in a table presented in [4–5, 7]. Note that a one to \( n-1 \) correspondence exists between the subgroups of \( H \) and the \( H \)-transitive \( n \)-colorings of the tiling of index \( n \).

With the complete orbifold presentation of subgroups and using the corresponding \( H \)-transitive colorings, we determine the subgroup fundamental regions. This enumeration of subgroups is constrained by considering a particular class of surfaces to reticulate. It is evident that the geometry and symmetry of these surfaces are related to the two-dimensional hyperbolic symmetries. Thus,
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making it possible to wrap hyperbolic tilings onto the triply periodic minimal surface. Tilings of triply periodic minimal surfaces may be lifted to the hyperbolic plane.

**H-transitive colorings of hyperbolic triangle tilings**

In generating 3D nets, we consider tilings of the hyperbolic plane whose group of symmetries G has at least distinct subgroups of symmetries of the surface. This correspondence, between the symmetries of the hyperbolic tiling and the minimal surface is the defined covering map. In particular, H-transitive colorings of the tiling associated with the subgroups of play G a vital role in the enumeration of the subgroup of symmetries.

We begin by constructing the hyperbolic triangle tiling. By repeatedly reflecting a triangle with interior angles \( \alpha \), \( \beta \), and \( \gamma \) in its sides, copies of the triangle cover the entire hyperbolic plane with no gaps and overlaps. The resulting tiling has the symmetry group generated by \( P \), \( Q \) and \( R \) satisfying the relation \( P^2 = Q^2 = R^2 = (PQ)^r = (QR)^p = (RP)^q \). This group is called the triangle group \(*_{pqr}\).

as an illustration, we present in Fig. 1 a triangle tiling by a fundamental triangle with interior angles \( \alpha \), \( \beta \), and \( \gamma \). In this case, \( H = *_{247} \).

Now, \( H \) acts transitively on the tiling. In generating subgroups of \( H \) of index \( n \), we construct all \( n \)-colorings of \( T \) using the set of \( n \) colors \( C \) where all elements of \( H \) effect permutations of \( C \) and \( H \) acts transitively on \( C \). In a H-transitive \( n \)-coloring of the tiling, the elements of \( H \) which fix a specific color in the colored tiling form a subgroup of \( H \) on index \( n \). Observe that corresponding to a subgroup \( S \) of \( H \) of index \( n \) there are \((n-1)!\) H-transitive \( n \)-colorings of the tiling that can be constructed with \( S \) fixing a particular color [4, 5].

To illustrate this derivation, consider Fig. 2. It shows a 2-coloring of the tiling by triangles. It is a result of a color scheme where each element \( P \), \( Q \), \( R \) interchanges colors \( c_1 \) and \( c_2 \). As a consequence of this assignment of colors, the rotations \( QR \), \( RP \) and \( PQ \) fix a specific color. Together, these rotations generate the index 2 subgroup fixing a specific color in the tiling.
For a given subgroup fixing a specific color in the tiling, the fundamental region is determined. The subgroup fundamental region is the union of the triangles in the original tiling such that by the subgroup action on the region it will cover the entire hyperbolic plane without gap and overlaps. Thus, this region is a compact 2D manifold. As an example, refer to Fig. 3a. The subgroup of index 2 fixing a specific color in the H-transitive coloring of the tiling has a fundamental region built from the two triangles in the tiling. The manifold is labelled by the symmetry of the subgroup using Conway's orbifold symbols. These symbols describe the symmetry operations within a fundamental region of the group, consisting of sites of rotational symmetry, labelled by digits followed by mirror paths, denoted by the string "*abc"... describing mirrors intersecting at angles of $\pi/a, \pi/b, \pi/c$, a glide reflections and translations labelled by x, o, respectively. The orbifold can be determined by specifying the number of handles or cross-caps (x), the cone points ("), boundary components (dashed / dark lines) and for each boundary component must list the branching numbers of all corner points lying on the component in cyclic order. In the colored pattern shown in Fig.2, the group of symmetries fixing a specific color is generated by the 2-fold rotation QR, 4-fold rotation RP and 7-fold rotation PQ. In Conway notation, this subgroup is denoted by 247 (vertices of the 247 pattern of Fig.2 are labelled according to their site symmetries). Its corresponding orbifold is shown in Fig. 3c.

Colored hyperbolic patterns in periodic minimal surfaces

We now consider the coloring associated to the subgroup of H. By taking the fundamental region of the subgroup fixing a specific color and applying the group action on the fundamental tile a new colored hyperbolic pattern is generated. Next is to glue together copies of the fundamental tile at the symmetrically equivalent edges and vertices of the pattern. Note that this gluing maps triangles of same color by the symmetry action of the subgroup. Let us illustrate this gluing...
On the use of colored hyperbolic subgroup patterns in finding covering maps for triply periodic minimal surfaces

process in the subgroup 247 shown in Fig. 4. This group is generated by the 2-, 4-, and 7-fold rotations. Copies of the quadrilateral are glued together at the vertices of the adjacent patterns. These vertices are fixed points of the equivalent rotational symmetries. This process results to a 3D topological surface called gyroid.

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REFERENCES